

An algebraically independent generating set of the algebra of local unitary invariants

Péter Vrana

Department of Theoretical Physics, Institute of Physics, Budapest University of
Technology and Economics, H-1111 Budapest, Hungary

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Abstract

We show that the inverse limit of the graded algebras of local unitary invariant polynomials of finite dimensional k -partite quantum systems is free, and give an algebraically independent generating set. The number of degree $2d$ invariants in the generating set is equal to the number of conjugacy classes of index d subgroups of a free group on $k - 1$ generators.

1 Introduction

One of the approaches to the problem of understanding quantum entanglement is to look for functions on the state space which are invariant under the action of the local unitary (LU) group and enable us to distinguish between different types of states. For a multipartite quantum system with distinguishable subsystems, this group is the product of the unitary groups acting on the Hilbert spaces of the individual subsystems.

The problem of separating the orbits can be reduced to finding the set of polynomial invariants [1], which form an algebra. Unfortunately, a description in terms of generators and relations is available only in the case of some special Hilbert space dimensions and particle numbers, in other cases, only partial results exist, see e.g. [2, 3, 4, 5].

In [6] it was pointed out that the dimension of LU-invariant homogenous polynomials with a fixed degree stabilizes as the dimensions of the Hilbert spaces of the subsystems increase. Based on this observation, one can introduce an algebra which can be thought of as gluing together the algebras of LU-invariant polynomials of various finite dimensional quantum systems [7], much like one studies the algebra of symmetric polynomials independently of the number of variables.

The outline of the paper is as follows. In section 2 we summarize the construction of the inverse limit of the algebras of LU-invariant polynomials over finite dimensional state spaces of quantum systems with a fixed number of subsystems. We call this object the algebra of local unitary invariants [7].

In section 3 we collect some facts about finite coverings of a graph. In particular, we describe a bijection between conjugacy classes of finite index subgroups of a free group, finite coverings of a certain graph and orbits of tuples of permutations under simultaneous conjugation following ref. [8].

In section 4 we prove that the algebra of local unitary invariants is free by giving an algebraically independent generating set. Our proof makes use of the invariants introduced in ref. [9].

Section 5 contains some concluding remarks, including the interpretation of our result in the context of LU-invariants of mixed states.

2 The algebra of local unitary invariants

Let $k \in \mathbb{N}$ and for every k -tuple $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ let us consider the complex Hilbert space $\mathcal{H}_n = \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ describing the pure states of a composite system with k distinguishable

subsystems. The group of local unitary transformations, $LU_n = U(n_1, \mathbb{C}) \times \cdots \times U(n_k, \mathbb{C})$, acts on \mathcal{H} in the obvious way, i.e. regarding \mathbb{C}^{n_i} as the standard representation of $U(n_i, \mathbb{C})$.

Let $I_{k,n}$ denote the algebra of LU_n -invariant polynomial functions over \mathcal{H}_n , regarded as a real vector space. Polynomial functions (with respect to any fixed basis) are in bijection with elements in $S(\mathcal{H}_n \oplus \mathcal{H}_n^*)$, the symmetric algebra on $\mathcal{H}_n \oplus \mathcal{H}_n^*$ on which an action of LU_n is induced and we have

$$I_{k,n} = S(\mathcal{H}_n \oplus \mathcal{H}_n^*)^{LU_n} \quad (1)$$

Note that in $I_{k,n}$ the polynomials are of the same degree in the coefficients and their complex conjugates, therefore we find it convenient to use a grading which is different from the usual one in a factor of two, and call homogenous degree m the polynomials which are of degree m both in the coefficients and their conjugates.

For $n \leq n' \in \mathbb{N}^k$ with respect to the componentwise order, we have the inclusion $\iota_{n,n'} : \mathcal{H}_n \hookrightarrow \mathcal{H}_{n'}$ which is the tensor product of the usual inclusions $\mathbb{C}^{n_i} \hookrightarrow \mathbb{C}^{n'_i}$ sending an n_i -tuple to the first n_i components. Similarly, we regard LU_n as a subgroup of $LU_{n'}$ which stabilizes the image of $\iota_{n,n'}$, and thus $\iota_{n,n'}$ is an LU_n -equivariant linear map, inducing a morphism of graded algebras $\varrho_{n,n'} : I_{k,n'} \rightarrow I_{k,n}$.

$((I_{k,n})_{n \in \mathbb{N}^k}, (\varrho_{n,n'})_{n \leq n' \in \mathbb{N}^k})$ is an inverse system of graded algebras, the inverse limit of which will be denoted by I_k and called the algebra of LU-invariants:

$$I_k := \varprojlim_{n \in \mathbb{N}^k} I_{k,n} = \left\{ (f_n)_{n \in \mathbb{N}^k} \in \prod_{n \in \mathbb{N}^k} I_{k,n} \mid \forall n \leq n' : f_n = \varrho_{n,n'} f_{n'} \right\} \quad (2)$$

Note that $I_{k,(n_1, \dots, n_k)}$ is a quotient of I_k and the restriction of the quotient map to the subspace of elements of degree at most $\min\{n_1, \dots, n_k\}$ is an isomorphism.

The dimension of the homogenous degree m subspace of I_k is given by

$$d_{k,m} = \sum_{a \vdash m} \left(\prod_{i=1}^m i^{a_i} a_i! \right)^{k-2} \quad (3)$$

and the Hilbert series of I_k is [7]

$$\sum_{m \geq 0} d_{k,m} t^m = \prod_{d \geq 1} (1 - t^d)^{-u_d(F_{k-1})} \quad (4)$$

where $u_d(F_{k-1})$ denotes the number of conjugacy classes of index d subgroups of F_{k-1} , the free group on $k-1$ generators.

Our aim is to prove that I_k is free, and the number of degree d invariants in an algebraically independent generating set equals the number of conjugacy classes of index d subgroups in the free group on $k-1$ generators, as the Hilbert series suggests.

3 Graph coverings

Let $G = (V, E)$ be a connected graph with coloured and directed edges (possibly multiple edges and/or loops). A graph $\tilde{G} = (\tilde{V}, \tilde{E})$ together with a projection $p : \tilde{G} \rightarrow G$ is said to be a covering of G if $p_V : \tilde{V} \rightarrow V$ and $p_E : \tilde{E} \rightarrow E$ are two surjections where the image of the head (tail) of an edge is the head (tail) of its image, p_E respects colours and such that the indegree and outdegree of every vertex $\tilde{v} \in \tilde{V}$ is the same as that of $p_V(\tilde{v})$ in each subgraph determined by the colours. A covering $p : \tilde{G} \rightarrow G$ is said to be finite if $|p_V^{-1}(v)| < \infty$ and m -fold if $|p_V^{-1}(v)| = m$ for all $v \in G$.

Two coverings $p_1 : \tilde{G}_1 \rightarrow G$ and $p_2 : \tilde{G}_2 \rightarrow G$ are said to be isomorphic if there exists an isomorphism $\varphi : \tilde{G}_1 \rightarrow \tilde{G}_2$ making the following diagram commute:

$$\begin{array}{ccc} \tilde{G}_1 & \xrightarrow{\varphi} & \tilde{G}_2 \\ p_1 \searrow & & \swarrow p_2 \\ & G & \end{array} \quad (5)$$

The set of isomorphism classes of finite coverings of a graph G will be denoted by $\mathbf{Iso}(G)$, the set of isomorphism classes of m -fold coverings by $\mathbf{Iso}(G, m)$, and the connected ones by $\mathbf{Isoc}(G)$ and $\mathbf{Isoc}(G, m)$, respectively.

Let G be the graph with a single vertex and $k - 1$ directed coloured loops. It is well-known that its fundamental group $\pi_1(G)$ is the free group of rank $k - 1$, and a set of generators may be identified with the directed loops. There exists a bijection between $\mathbf{Isoc}(G, m)$ and conjugacy classes of subgroups of index m of $\pi_1(G)$.

Let S_m denote the group of bijections from $\{1, \dots, m\}$ to itself. This group acts on $S_m^{k-1} = S_m \times S_m \times \dots \times S_m$ by simultaneous conjugation:

$$\pi \cdot (\sigma_1, \dots, \sigma_{k-1}) = (\pi \sigma_1 \pi^{-1}, \dots, \pi \sigma_{k-1} \pi^{-1}) \quad (6)$$

Let us denote the orbit of $(\sigma_1, \dots, \sigma_{k-1}) \in S_m^{k-1}$ by

$$[\sigma_1, \dots, \sigma_{k-1}] = \{\pi \cdot (\sigma_1, \dots, \sigma_{k-1}) \mid \pi \in S_m\} \quad (7)$$

To a (not necessarily connected) m -fold covering \tilde{G} of G we can associate an element in

$$S_m^{k-1}/S_m = \{[\sigma_1, \dots, \sigma_{k-1}] \mid \forall i : \sigma_i \in S_m\} \quad (8)$$

as follows. Label the vertices of \tilde{G} arbitrarily with the numbers $\{1, \dots, m\}$ using each label exactly once. Let σ_i be the permutation which sends a to b if there is a directed edge from a to b of colour i in \tilde{G} . Note that this gives indeed a $k - 1$ -tuple of permutations as the indegree and outdegree of every vertex in \tilde{G} is 1 in the subgraph determined by any colour. As relabelling corresponds to simultaneous conjugation, we have indeed a well-defined map $\Phi : \mathbf{Iso}(G, m) \rightarrow S_m^{k-1}/S_m$.

Now let \tilde{G}_1 and \tilde{G}_2 be two coverings of G where \tilde{G}_1 is m_1 -fold and \tilde{G}_2 is m_2 -fold. The disjoint union $\tilde{G} := \tilde{G}_1 \sqcup \tilde{G}_2$ is then an $m_1 + m_2$ -fold covering of G . We would like to relate the orbits of $k - 1$ -tuples of permutations of the three coverings. Let us choose the numbering of the vertices of \tilde{G} so that \tilde{G}_1 is labelled with $\{1, \dots, m_1\}$ and \tilde{G}_2 is labelled with $\{m_1 + 1, \dots, m_1 + m_2\}$.

Let $(\sigma_1^{(j)}, \dots, \sigma_{r-1}^{(j)})$ be the representative of the orbit corresponding to \tilde{G}_j and $(\sigma_1, \dots, \sigma_{r-1})$ be that of \tilde{G} which can be read off from the above-chosen labelling (after subtracting m_1 in the $j = 2$ case). It is easy to see that for all $1 \leq i \leq k - 1$ we have

$$\sigma_i(a) = \begin{cases} \sigma_i^{(1)}(a) & \text{if } a \leq m_1 \\ \sigma_i^{(2)}(a - m_1) + m_1 & \text{if } a > m_1 \end{cases} \quad (9)$$

given by the usual homomorphism $S_{m_1} \times S_{m_2} \hookrightarrow S_{m_1+m_2}$. This map clearly induces a map $\star : S_{m_1}^{k-1}/S_{m_1} \times S_{m_2}^{k-1}/S_{m_2} \rightarrow S_{m_1+m_2}^{k-1}/S_{m_1+m_2}$ on the orbits (we will use infix notation, i.e. the map sends $(a, b) \mapsto a \star b$). One can see immediately that \star turns the set $\bigsqcup_{m=1}^{\infty} S_m^{k-1}/S_m$ into a commutative semigroup. Also, $\mathbf{Iso}(G)$ can be equipped with a semigroup structure induced by disjoint union, and Φ is an isomorphism.

4 Algebraically independent generators of the algebra of LU-invariants

To an orbit in S_m^{k-1}/S_m we can associate an element in I_k as follows. It was shown in ref. [7] that every element of I_k is represented in some $I_{k,n}$. We will give a representative in $I_{k,n}$ where $n = (n_1, \dots, n_k) \geq (m, \dots, m)$ following ref. [9]. A vector in \mathcal{H}_n is of the form

$$\psi = \sum_{i_1, \dots, i_k} \psi_{i_1, \dots, i_k} e_{i_1} \otimes \dots \otimes e_{i_k} \quad (10)$$

where in the sum $1 \leq i_j \leq n_j$ for all $1 \leq j \leq k$.

Let $(\sigma_1, \dots, \sigma_{k-1}) \in S_m^{k-1}$ be a representative. The value of the associated polynomial on ψ is

$$f_{[\sigma_1, \dots, \sigma_{k-1}]}(\psi) = \sum_{i_1^1, \dots, i_k^m} \psi_{i_1^1, \dots, i_k^1} \dots \psi_{i_1^m, \dots, i_k^m} \overline{\psi_{i_1^{\sigma_1(1)}, \dots, i_{k-1}^{\sigma_{k-1}(1)}, i_k^1}} \dots \overline{\psi_{i_1^{\sigma_1(m)}, \dots, i_{k-1}^{\sigma_{k-1}(m)}, i_k^m}} \quad (11)$$

where the sum is over all $k \cdot m$ -tuples of integers where $1 \leq i_j^l \leq n_j$ for all $1 \leq j \leq k$ and $1 \leq l \leq m$. Note that the expression defining $f_{[\sigma_1, \dots, \sigma_{k-1}]}$ is independent of the choice of the representative, justifying the notation.

An important observation is the following:

Lemma 1. *Let $[\sigma_1^{(1)}, \dots, \sigma_{k-1}^{(1)}] \in S_{m_1}^{k-1}/S_{m_1}$ and $[\sigma_1^{(2)}, \dots, \sigma_{k-1}^{(2)}] \in S_{m_2}^{k-1}/S_{m_2}$. Then*

$$f_{[\sigma_1^{(1)}, \dots, \sigma_{k-1}^{(1)}] \star [\sigma_1^{(2)}, \dots, \sigma_{k-1}^{(2)}]} = f_{[\sigma_1^{(1)}, \dots, \sigma_{k-1}^{(1)}]} f_{[\sigma_1^{(2)}, \dots, \sigma_{k-1}^{(2)}]} \quad (12)$$

Proof. Let the representative of the orbit $[\sigma_1^{(1)}, \dots, \sigma_{k-1}^{(1)}] \star [\sigma_1^{(2)}, \dots, \sigma_{k-1}^{(2)}]$ given by eq. (9) be $(\sigma_1, \dots, \sigma_{k-1}) \in S_{m_1+m_2}^{k-1}$ and let us denote $m_1 + m_2$ by m . Then

$$\begin{aligned} & \sum_{i_1^1, \dots, i_k^m} \psi_{i_1^1, \dots, i_k^1} \dots \psi_{i_1^m, \dots, i_k^m} \overline{\psi_{i_1^{\sigma_1(1)}, \dots, i_{k-1}^{\sigma_{k-1}(1)}, i_k^1}} \dots \overline{\psi_{i_1^{\sigma_1(m)}, \dots, i_{k-1}^{\sigma_{k-1}(m)}, i_k^m}} \\ &= \sum_{i_1^1, \dots, i_k^{m_1}} \sum_{i_1^{m_1+1}, \dots, i_k^m} \psi_{i_1^1, \dots, i_k^1} \dots \psi_{i_1^{m_1}, \dots, i_k^{m_1}} \overline{\psi_{i_1^{\sigma_1(1)}, \dots, i_{k-1}^{\sigma_{k-1}(1)}, i_k^1}} \dots \overline{\psi_{i_1^{\sigma_1(m_1)}, \dots, i_{k-1}^{\sigma_{k-1}(m_1)}, i_k^{m_1}}} \\ &= \sum_{i_1^1, \dots, i_k^{m_1}} \psi_{i_1^1, \dots, i_k^1} \dots \psi_{i_1^{m_1}, \dots, i_k^{m_1}} \overline{\psi_{i_1^{\sigma_1^{(1)}(1)}, \dots, i_{k-1}^{\sigma_{k-1}^{(1)}(1)}, i_k^1}} \dots \overline{\psi_{i_1^{\sigma_1^{(1)}(m_1)}, \dots, i_{k-1}^{\sigma_{k-1}^{(1)}(m_1)}, i_k^{m_1}}} \\ & \cdot \sum_{i_1^1, \dots, i_k^{m_2}} \psi_{i_1^1, \dots, i_k^1} \dots \psi_{i_1^{m_2}, \dots, i_k^{m_2}} \overline{\psi_{i_1^{\sigma_1^{(2)}(1)}, \dots, i_{k-1}^{\sigma_{k-1}^{(2)}(1)}, i_k^1}} \dots \overline{\psi_{i_1^{\sigma_1^{(2)}(m_2)}, \dots, i_{k-1}^{\sigma_{k-1}^{(2)}(m_2)}, i_k^{m_2}}} \end{aligned} \quad (13)$$

□

In other words, the map $\bigsqcup_{m=1}^{\infty} S_m^{k-1}/S_m \rightarrow I_k$ given by $s \mapsto f_s$ is a semigroup-homomorphism. Now we are ready to prove our main theorem:

Theorem 2. *I_k is freely generated by the set*

$$F := \{f_{\Phi(\tilde{G})} | \tilde{G} \in \mathbf{Isoc}(G)\} \quad (14)$$

Proof. In ref. [9] it was shown that the set

$$\{f_s | s \in S_m^{k-1}/S_m\} = \{f_{\Phi(\tilde{G})} | \tilde{G} \in \mathbf{Iso}(G, m)\} \quad (15)$$

forms a basis of the degree m homogenous subspace of $I_{k,n}$ (when represented as polynomials) if $n \geq (m, \dots, m)$. Therefore, it is also a basis of the degree m homogenous subspace of I_k . As I_k is the direct sum of its homogenous subspaces, we conclude that $\{f_{\Phi(\tilde{G})} | \tilde{G} \in \mathbf{Iso}(G)\}$ is a basis of I_k . Note that this also implies that the map $\tilde{G} \mapsto f_{\Phi(\tilde{G})}$ is injective.

An element of the form f_s where $s \in S_m^{k-1}/S_m$ can be uniquely written as the product of some elements of F . Indeed, $\Phi^{-1}(s)$ is a covering of G , which can be uniquely written as a disjoint union of connected coverings $\tilde{G}_1, \dots, \tilde{G}_d$ (up to isomorphism and ordering), and therefore

$$f_s = f_{\Phi(\tilde{G}_1 \sqcup \dots \sqcup \tilde{G}_d)} = f_{\Phi(\tilde{G}_1) \star \dots \star \Phi(\tilde{G}_d)} = f_{\Phi(\tilde{G}_1)} \cdots f_{\Phi(\tilde{G}_d)} \quad (16)$$

□

5 Conclusion

We have shown that the inverse limit I_k of the algebras of LU-invariant polynomials of pure states of k -partite quantum systems with finite dimensional Hilbert spaces is free, and an algebraically independent generating set can be given in terms of finite connected coverings of a graph with a single vertex and k loops. The number of homogenous degree $2d$ polynomials in the algebraically independent generating set equals the number of isomorphism classes of d -fold connected coverings, which in turn equals the number of conjugacy classes of index d subgroups of a free group on $k-1$ generators.

In light of the close relationship between LU-equivalence classes of mixed states of a k -particle quantum system and those of pure states of a $k+1$ -particle quantum system [10, 7], one should be able to interpret our result in the context of mixed states. This can be done as follows.

Observe that each term on the right hand side of eq. (11) depends only on the reduced density matrix obtained when we trace over the last subsystem. Therefore it is easy to translate the result to the case of mixed state local unitary invariants. Let I_k^{mixed} denote the inverse limit of the algebras of LU-invariants of mixed states over k -partite quantum systems with finite dimensional Hilbert space as in [7]. Let G be the graph with a single vertex and k directed labelled edges. To a connected covering $\tilde{G} \in \mathbf{Iso}(G, m)$ we associate the following invariant with $[\sigma_1, \dots, \sigma_k] = \Phi(\tilde{G})$.

$$f_{[\sigma_1, \dots, \sigma_k]}(\varrho) = \sum_{i_1^1, \dots, i_k^m} \varrho_{i_1^1, \dots, i_k^1, i_1^{\sigma_1(1)}, \dots, i_k^{\sigma_k(1)}} \cdots \varrho_{i_1^m, \dots, i_k^m, i_1^{\sigma_1(m)}, \dots, i_k^{\sigma_k(m)}} \quad (17)$$

where

$$\varrho = \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \varrho_{i_1, \dots, i_k, j_1, \dots, j_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_k}^* \quad (18)$$

is an arbitrary mixed state.

Explicite descriptions of the algebras $I_{k,n}$ are known in only a limited number cases, including $k=2$, n arbitrary, $k=3$, $n=(2,2,2)$ [1] and $k=4$, $n=(2,2,2,2)$ [11]. It should be noted that for any $k \in \mathbb{N}$ and $n \in \mathbb{N}^k$, $I_{k,n}$ is a quotient of I_k . It would be interesting to determine the kernels of the quotient maps in each case.

We would like to emphasize that in spite of our lack of knowledge about the structure of every single $I_{k,n}$, fortunately the generators of I_k can be directly interpreted as generators of the algebras of invariants of pure states of arbitrary k -partite quantum systems.

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